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1992 J. Phys. A: Math. Gen. 25 L409

(http://iopscience.iop.org/0305-4470/25/8/005)

J. Phys. A: Math. Gen. 25 (1992) L409–L412. Printed in the UK

LETTER TO THE EDITOR

Extremal trajectories for stochastic equations obtained directly from the Langevin differential operator: II. First integrals

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Received 29 November 1991

Abstract. We have shown that the differential operator for the extremal trajectory of a stochastic process can be written as a *square* of operators, i.e. the Langevin systematic operator times its adjoint. Here we show that we can go further and write also directly from the Langevin equation, first integrals (conservation principles) of the extremal path differential equation. We assume linearity in the Langevin operator and Gaussianity for the fluctuation.

Following the functional integral approach, which now has been applied to stochastic problems by several authors [2–5], when the integrals involved are Gaussian, then the conditional probability density can be expressed as an exponential of a functional $S[x(t)]$ evaluated along an extremal path. The calculation of this extremal path usually requires a variational process that gives us a differential equation for the path.

We obtained recently a result [1] that allows us to write the extremal path differential operator directly in terms of products associated with the Langevin operator. Here we want first to recall that result in a more compact expression and then, going further, we obtain first integrals of the extremal path differential equation.

We start with a general stochastic equation with additive and Gaussian noise

$$D(x)=f(t) \tag{1}$$

where we define a linear differential operator, which hereafter is referred to as the Langevin operator, as

$$D=a_N\frac{\mathrm{d}^{(N)}}{\mathrm{d}t^N}+a_{N-1}\frac{\mathrm{d}^{(N-1)}}{\mathrm{d}t^{N-1}}+\ldots+a_0 \tag{2}$$

with a_j analytic functions of t.

$f(t)$ is a stochastic function with a correlation

$$\langle f(t_1)f(t_2)\rangle=(D/\tau)\exp(-|t_1-t_2|/\tau) \tag{3}$$

where τ is the correlation time; then we can write together with equation (1), the equation

$$\tau\frac{\mathrm{d}f}{\mathrm{d}t}=-f+\xi \tag{4}$$

with ξ being delta correlated. Taking the derivative of equation (1) we obtain

$$\left(1+\tau\frac{\mathrm{d}}{\mathrm{d}t}\right)Dx=\xi(t). \tag{5}$$

Now, we define the 'action' as the functional [6]

$$S[x(t)]=\int_{t_1}^{t_2} L(t)\,\mathrm{d}t \tag{6}$$

where

$$L(t)=\xi(t)^2 \tag{7}$$

because ξ is Gaussian and delta correlated. Then we have

$$L(t)=\left[\left(1+\tau\frac{\mathrm{d}}{\mathrm{d}t}\right)Dx\right]^2. \tag{8}$$

Applying the stationary condition $\delta S=0$ to equation (6) and after several integrations by parts one obtains

$$\sum_{k=0}^{N+1}(-1)^k\frac{\mathrm{d}^{(k)}}{\mathrm{d}t^k}\frac{\partial L}{\partial x^{(k)}}=0 \tag{9}$$

which is the Euler–Lagrange differential equation for the extremal path.

Now, from equations (2), (8) and (9) we obtain

$$D^*\left(1-\tau^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)Dx=0 \tag{10}$$

where D^* stands for the adjoint of D

$$D^*=\sum_{k=0}^{N}(-1)^k\frac{\mathrm{d}^{(k)}}{\mathrm{d}t^k}a_{N-k}. \tag{11}$$

We can write equation (10) in the form†

$$D^*M^*MD(x)=0 \tag{12}$$

for which we define the *memory* operator $M=1+\tau\,\mathrm{d}/\mathrm{d}t$ and its adjoint $M^*=1-\tau\,\mathrm{d}/\mathrm{d}t$.

First we observe in equation (12) that the whole operator for the extremal is self-adjoint; we will come back to this property later. We see in this product an operator times its adjoint; this square comes from the Gaussian noise. If the coefficients of D are not constant then the operator M and D do not commute and this factorization can be useful to solve a complicated differential equation. When the coefficients of D are constant then the four operators of equation (12) commute and we can write the solution of the extremal path as a linear combination of the solutions of $D(x)=0$, $D^*(x)=0$ and the functions $\exp(t/\tau)$ and $\exp(-t/\tau)$. These exponentials are the solutions that correspond to the memory operators and they give to the extremal path the memory contribution, no matter the precise form of the Langevin operator D.

We observe in equation (12) that if the correlation time τ goes to zero then the equation reduces to $D^*D(x)=0$, which is the Markovian limit.

† In a previous communication we got this result with the factors in a different order and that expression is more complicated due to the commutation relations.

Now let us consider the converse problem. We have that, given any self-adjoint differential operator R, if it is of even order, it can be factorized in the form [7]

$$R = D^* D \tag{13}$$

then we will see how the function $D(x)$ that results can be related to first integrals of the differential equation $R(x) = 0$.

To illustrate this let us consider a simple dynamical non-stochastic problem, starting from an equation of motion whose differential operator has the self-adjoint property. Let us take the equation of the harmonic oscillator,

$$R(x) = \ddot{x} + \omega^2 x = 0. \tag{14}$$

This operator can be factorized as

$$R = \left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\omega\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{i}\omega\right). \tag{15}$$

We define $D = (\mathrm{d}/\mathrm{d}t) + \mathrm{i}\omega$ and $D^* = (\mathrm{d}/\mathrm{d}t) - \mathrm{i}\omega$. Then we can define a Lagrangian† as $L = D(x)^2$. In this case D has constant coefficients, therefore L does not depend explicitly on t. Then we can identify [8] a first integral of the equation $R(x) = 0$ as

$$H = \dot{x}\frac{\partial L}{\partial \dot{x}} - L \tag{16}$$

which in terms of D gives

$$H = D(x)[D(x) - 2\mathrm{i}\omega x] = \dot{x}^2 + \omega^2 x^2 \tag{17}$$

which means of course the energy conservation.

Now we go back to the Langevin equation (1); let us consider a general case where D is linear and we want to identify first integrals that should arise: (a) when D does not depend explicitly on time and (b) when D does not depend explicitly on x.

(a) Suppose D has constant coefficients. Starting from the expression (2) and taking the Lagrangian as

$$L = D(x)^2 \tag{18}$$

we can write a general expression for the first integral of the extremal differential equation [8]

$$\begin{aligned} H = x^{(1)}&\left[\frac{\partial L}{\partial x^{(1)}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial x^{(2)}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial x^{(3)}} + \ldots + (-1)^{N-1}\frac{\mathrm{d}^{(N-1)}}{\mathrm{d}t^{N-1}}\frac{\partial L}{\partial x^{(N)}}\right] \\ &+ x^{(2)}\left[\frac{\partial L}{\partial x^{(2)}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial x^{(3)}} + \ldots + (-1)^{N}\frac{\mathrm{d}^{(N-2)}}{\mathrm{d}t^{N-2}}\frac{\partial L}{\partial x^{(N)}}\right] \\ &+ x^{(3)}\left[\frac{\partial L}{\partial x^{(3)}} - \ldots + (-1)^{N-1}\frac{\mathrm{d}^{(N-3)}}{\mathrm{d}t^{N-3}}\frac{\partial L}{\partial x^{(N)}}\right] \\ &+ \ldots + x^{(N)}\frac{\partial L}{\partial x^{(N)}} - L \end{aligned} \tag{19}$$

† We note here that the Lagrangian defined as $D(x)^2$, in this case, turns out to be a complex quantity and has a term in $x\dot{x}$. We know that, given an Euler–Lagrange equation, the corresponding Lagrangian is not unique.

(here the exponentials in parentheses mean time derivatives) and from equation (18), we can have this expression in terms of D:

$$H = D^2 - 2xa_0 D - 2x^{(1)}[a_2 D^{(1)} - a_3 D^{(2)} + \ldots + (-1)^N a_N D^{(N-1)}]$$
$$- 2x^{(2)}[a_3 D^{(1)} - \ldots + (-1)^{(N-1)} a_N D^{(N-2)}] - \ldots - 2x^{(N-1)} a_N D^{(1)}. \qquad (20)$$

We point out that here D is not an operator, but a function of $x, \dot{x}, \ddot{x}$, etc.

(*b*) Now suppose that D can have variable coefficients but does not depend explicitly on x (an ignorable variable). From the Euler–Lagrange equation (extremal equation), we can integrate once and write this integral in terms of D as

$$Da_1 - \frac{d}{dt}(Da_2) + \frac{d^2}{dt^2}(Da_3) - \frac{d^3}{dt^3}(Da_4) + \ldots + (-1)^{N-1}\frac{d^{(N-1)}}{dt^{N-1}}(Da_N) = C \qquad (21)$$

or we could write this expression in the form

$$Q^* D(x) = Q^* \frac{dQ}{dt}(x) = C \qquad (22)$$

where we define the operator Q associated with D as

$$Q = a_1 + a_2 \frac{d}{dt} + \ldots + a_N \frac{d^{(N-1)}}{dt^{N-1}}. \qquad (23)$$

We see that the outcoming operator $Q^*\, dQ/dt$ is also self-adjoint, of order $2N-1$.

We have obtained in equations (12), (20) and (22) some results about extremal paths and conservation principles that depend on the Langevin operator, and the nature of noise. So far, these expressions which involve differential equations and a variational principle have to be constrained to linear operators. We are trying to consider nonlinear Langevin operators that could include more complicated potentials. The difficulties one finds, depending on the particular problem, are not only to deal with nonlinear operators, but also that the functional integrals are no longer Gaussian and the extremal path may not be the only contribution to the sum over paths.

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